Math 010 Exam 1
Spring 2026

For full credit: Please show work using techniques from this course and use correct mathematical notation.

1. (7 pts) Determine conditions (if any) that a, b, and c must satisfy for the linear system to be consistent.

$$\begin{aligned} x_1 + 3x_2 - x_3 &= a \\ 2x_1 + 5x_2 + x_3 &= b \\ 3x_1 + 8x_2 &= c \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & a \\ 2 & 5 & 1 & b \\ 3 & 8 & 0 & c \end{array}\right]$$

$R_2 \Rightarrow R_2 - 2R_1$

$$\begin{array}{cccc} 2 & 5 & 1 & b \\ -2 & -6 & 2 & -2a \\ \hline 0 & -1 & 3 & b-2a \end{array}$$

$R_3 \Rightarrow R_3 - 3R_1$

$$\begin{array}{cccc} 3 & 8 & 0 & c \\ -3 & -9 & 3 & -3a \\ \hline 0 & -1 & 3 & c-3a \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & a \\ 0 & -1 & 3 & b-2a \\ 0 & -1 & 3 & c-3a \end{array}\right]$$

$R_3 \Rightarrow R_3 - R_2$

$$\begin{array}{cccc} 0 & -1 & 3 & c-3a \\ 0 & 1 & -3 & -b+2a \\ \hline 0 & 0 & 0 & c-b-a \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & a \\ 0 & -1 & 3 & b-2a \\ 0 & 0 & 0 & c-b-a \end{array}\right] \rightarrow$$ The system is consistent if $\boxed{c = a+b}$.

2. (6 pts) The augmented matrix for a linear system of equations has been reduced to reduced row echelon form. Express the solution set as a linear combination of column vectors that contain only numerical entries.

$$\left[\begin{array}{ccccc|c} 1 & -2 & 0 & 3 & 0 & 4 \\ 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 6 \end{array}\right]$$

$R_1: x_1 = 2x_2 - 3x_4 + 4$

$R_2: x_3 = -x_4 - 1$

$R_3: x_5 = 6$

Let $s = x_2$ and $t = x_4$.

Then $\vec{x} = \begin{bmatrix} 2s-3t+4 \\ s \\ -t-1 \\ t \\ 6 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 4 \\ 0 \\ -1 \\ 0 \\ 6 \end{bmatrix}}$

3. Let $A = \begin{bmatrix} 2 & -1 \\ 0 & 3 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & -3 \\ 8 & 1 \end{bmatrix}$. Compute the indicated expression or say why the operation is not defined.

a. (3 pts) AB

$$\begin{bmatrix} 2 & -1 \\ 0 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-3 & 4+0 & -2-1 \\ 0+9 & 0+0 & 0+3 \\ 1+6 & 2+0 & -1+2 \end{bmatrix} = \begin{bmatrix} -1 & 4 & -3 \\ 9 & 0 & 3 \\ 7 & 2 & 1 \end{bmatrix}$$

b. (3 pts) $tr(BA)$

$$BA = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 0 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2+0-1 & -1+6-2 \\ 6+0+1 & -3+0+2 \end{bmatrix}$$

$$tr(BA) = \underline{1-1} = 0$$

OR $tr(BA) = tr(AB) = 0$ from part (a).

c. (2 pts) $AC + B$

AC is $(3 \times 2) \times (2 \times 2)$ and so is 3×2.

B is 2×3. Since AC and B are not the same size, the operation is not defined.

4. a. (4 pts) Let $A = \begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix}$. Find A^{-1}.

$\det(A) = 8 - 7 = 1$

$A^{-1} = \begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix}$

b. (2 pts) Use your answer from part (a) to solve the linear system.

$4x_1 + 7x_2 = 3$
$x_1 + 2x_2 = 1$

$\begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6-7 \\ -3+4 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

5. (8 pts) Use the inversion algorithm to determine whether A is invertible or singular. You don't need to find the inverse; stop working when you can answer the question with certainty.

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \Rightarrow R_1 - 2R_2$

$$\begin{array}{cccccc} 2 & 1 & 0 & 1 & 0 & 0 \\ -2 & -6 & -1 & 0 & -2 & 0 \\ \hline 0 & -5 & -1 & 1 & -2 & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & -5 & -1 & 1 & -2 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$R_3 \Rightarrow R_2 + 5R_3$

$$\begin{array}{cccccc} 0 & -5 & -1 & 1 & -2 & 0 \\ 0 & 5 & 10 & 0 & 0 & 5 \\ \hline 0 & 0 & 9 & 1 & -2 & 5 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & -5 & -1 & 1 & -2 & 0 \\ 0 & 0 & 9 & 1 & -2 & 5 \end{array}\right]$$

Since every row has a pivot, A is invertible.

6. a. (4 pts) Given $B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 3 \\ -8 & -4 & 1 \end{bmatrix}$, find an elementary matrix E such that $EB = A$.

R_3 of A is $-4R_1 + R_3$ of B

Then $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$.

b. (2 pts) Compute the product EB to verify your answer.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 3 \\ -8 & -4 & 1 \end{bmatrix} = A \checkmark$$

7. (3 pts) Let $M = \begin{bmatrix} 7 & 2a-1 \\ a+4 & b \end{bmatrix}$. Find all values of a and b so that M is a symmetric matrix.

b is any number.

$a+4 = 2a-1 \Rightarrow a=5$

8. Determine whether each statement is true or false and justify your answer (the justification can be one or two sentences or a counterexample, as appropriate).

a. (3 pts) If the reduced row echelon form of the augmented matrix for a linear system has a row of zeros, then the system must have infinitely many solutions.

False. This system has no solution:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

b. (3 pts) A homogenous linear system with more unknowns than equations always has infinitely many solutions.

True. Then its solution contains at least one parameter. This happens because if there are more rows (equations) than columns (variables) in the matrix, not every column has a leading 1.

9. (8 pts) Prove that if B is obtained from A by performing a sequence of elementary row operations, then there is a second sequence of elementary row operations, which when applied to B recovers A.

If A & B are as described, then there is a sequence of elementary row operations represented by $E_1, E_2, \dots, E_r$ such that $E_r \cdots E_2 E_1 A = B$.

Since each E_i is invertible, so is their product. Then

$$A = (E_r \cdots E_2 E_1)^{-1} B = E_1^{-1} E_2^{-1} \cdots E_r^{-1} B.$$

Each E_i^{-1} represants a row operation, so the sequence of row operations applied to B gives A.

10. (8 pts) Prove that if A and B are invertible $n \times n$ matrices, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

$$\begin{aligned}(B^{-1}A^{-1})(AB) &= B^{-1}(A^{-1}A)B \\ &= B^{-1} I B \\ &= B^{-1} B \\ &= I.\end{aligned}$$

Since when AB is multiplied by $B^{-1}A^{-1}$, the result is I,

$$B^{-1}A^{-1} = (AB)^{-1}.$$